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Can we find a new deformation of (SL_J)?
— Conjectures and supporting evidences —

by

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§0. Background of the problem.

In developing WKB analysis of Painlevé transcendents, we have introduced a large parameter η into the Painlevé equation (P_J) $(J = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VI})$ and the associated Schrödinger equation (SL_J) ([KT1]); the parameter η is designed to introduce a filtration suited for our study ([KT1], [AKT], [KT2], [KT3]). At the same time some special class of solutions of the equation (P_J) which have some homogeneity property with respect to η is sometimes important, particularly when we apply our results to the study of ordinary Painlevé transcendents (i.e., with η being set to be 1). As we note in Section 1, an important class of homogeneous solutions is given by what we call *pure solutions* of (P_J), that is, multiple-scale solutions obtained by setting all α_j and β_j to be 0 except for α_0 and β_0 in the construction described in Section 1 of [AKT]. As is explained in Section 1, the homogeneity is attained by attributing some homogeneity degree not only to the independent variables of (SL_J) (i.e., x and t) but also to relevant parameters (such as α in (P_{II})). In connection with this fact, it is an important issue to show that the invariant $E_J(\eta) = \sum_{j\geq 0} E_j\eta^{-j}$ of (SL_J) near the double turning point (cf. [AKT, §3]) is independent of the parameters contained in the coefficients of (P_J). As one can easily see (cf. Proposition 1.1 in Section 1), this property of E_J is equivalent to the assertion that E_j vanishes except for E_0 if the relevant 2-parameter solution $\lambda_J(t;\alpha,\beta)$ is pure. This assertion "$E_j = 0$ $(j \neq 0)$" is a quite intriguing one, and it is rather hard to believe it. But a computer-assisted computation done by T. Aoki really validates the vanishing of E_j $(j = 1, 2)$ in the case of (P_{II}). This result of Aoki is very remarkable and encouraging.

Now let us recall that the constancy of E_J in t is a consequence of the fact that (SL_J) is isomonodromically deformed in t ([KT2]). We then wonder the invariance of E_J with respect to the parameters in the coefficients of (P_J) might be a symptom of the (hitherto unknown) deformation in the parameters.

The principal aim of this report is to construct explicitly a candidate for a new differential equation which is compatible with (SL_J) when $J = \mathrm{II}$.

The notations and symbols we use here are mostly the same as those in [AKT] and [KT3]. Throughout this report we basically concentrate our attention to the case $J = \mathrm{II}$, although most of the formal aspects of the problem are uniformly valid for all J. We also use the symbol a to denote the parameter α in (P_{II}) (to distinguish it from the parameter α in $\lambda_{\mathrm{II}}(t;\alpha,\beta)$).

§1. Preliminaries.

In [AKT, §1 and Appendix] functions $\phi_J(t)$ and $\theta_J(t)$, which are basic ones in our construction of solutions of (P_J), have been given in a form involving indefinite integrals. There is, however, a uniform way of normalizing these functions without ambiguities resulting ¿from the choice of the end-points of the integrals. We refer the reader to [KT3] and [T] for the uniform description of ϕ_J and θ_J for general J, and here we tabulate the normalized ϕ_J and θ_J for $J = \mathrm{I}$ and II.

Table 1.1.

(i)
$$\phi_{\mathrm{I}}(t) = \int_0^t \sqrt{12\lambda_0(t)}dt$$
and
$$\theta_{\mathrm{I}}(t) = 2^2 3^5 \lambda_0(t)^5 \eta^2 \quad \text{with } 6\lambda_0^2 + t = 0.$$

(ii)
$$\phi_{\mathrm{II}}(t) = \int_r^t \sqrt{6\lambda_0^2 + tdt}$$
and
$$\theta_{\mathrm{II}}(t) = \frac{(6\lambda_0^2+t)^5\lambda_0^2\eta^2}{2^4 a^2},$$
where $2\lambda_0^3 + t\lambda_0 + a = 0$ and $6\lambda_0(r)^2 + r = 0$.

The extra factor η^2 in θ_{I} and θ_{II} above makes pure multiple-scale solutions to be homogeneous with respect to η, that is, $\lambda_{\mathrm{I}}\,(t;\alpha_0,\beta_0)$ (resp., $\lambda_{\mathrm{II}}\,(t,a;\alpha_0,\beta_0)$) assumes the form $\eta^{-2/5}f(\eta^{4/5}t)$ (resp., $\eta^{-1/3}g(\eta^{2/3}t,\eta a)$). (See [KT3] and [T] for the details.) Another important consequence of the uniform normalization used here is that the correspondence between free parameters contained in the multiple-scale solutions (such as (4.49) and (4.50) in [AKT]) becomes quite simple. See [T] for the details; we only note that (4.49) and (4.50) in [AKT] become simply $\alpha_0 = \tilde{\alpha}_0$ and $\beta_0 = \tilde{\beta}_0$ under this uniform normalization.

The homogeneity of the pure multiple-scale solution $\lambda_{\mathrm{II}}(t,a;\alpha_0,\beta_0)$ mentioned above makes $S_{\mathrm{odd}}(x,t,a;\eta)$ for (SL_{II}) to enjoy the following homogeneity property:

$$S_{\mathrm{odd}}(r^{-1/3}x, r^{-2/3}t, r^{-1}a, r\eta) = r^{1/3}S_{\mathrm{odd}}\ (x,t,a,\eta) \quad \text{for any } r>0 \quad , \tag{1.1}$$

or equivalently,

$$\Big(-\frac{2}{3}t\frac{\partial}{\partial t} - \frac{1}{3}x\frac{\partial}{\partial x} - a\frac{\partial}{\partial a} + \eta\frac{\partial}{\partial \eta} - \frac{1}{3}\Big)S_{\mathrm{odd}}\ (x,t,a,\eta) = 0. \tag{1.2}$$

We then use the deformation equation

$$\frac{\partial}{\partial t}S_{\mathrm{odd}} = \frac{\partial}{\partial x}(A_{\mathrm{II}}S_{\mathrm{odd}}) \tag{1.3}$$

to derive the following relation (1.4) from (1.2):

$$\eta\frac{\partial}{\partial \eta}\oint_{C(\lambda_0)} S_{\mathrm{odd}}dx = a\frac{\partial}{\partial a}\oint_{C(\lambda_0)} S_{\mathrm{odd}}\ dx, \tag{1.4}$$

where $C(\lambda_0)$ designates a sufficiently tiny circle around $x=\lambda_0(t)$. Here we have also used the relation $(x\partial/\partial x+1)S_{\mathrm{odd}} = (\partial/\partial x)(xS_{\mathrm{odd}})$.

Let us now recall that the invariant $E(\eta)$ is equal to

$$\frac{2}{\pi i}\oint_{C(\lambda_0)} S_{\mathrm{odd}}\ dx. \tag{1.5}$$

(Cf. [AKT, §3]) Thus the reletion (1.4) entails the following

Proposition 1.1. *The following assertions are equivalent:*
(A.1) *$E(\eta)$ is a genuine constant, that is, $E(\eta)$ consists of one term E_0.*
(A.2) *$E(\eta)$ is independent of a, that is, $\partial E/\partial a$ vanishes identically.*

Our hope is that the verification of (A.2) might be analytically more amenable than (A.1). In what follows we say for brevity that a formal series u in $\eta^{-1/2}$ is single-valued (near λ_0) if all of its coefficients are single-valued analytic functions with possible pole singularities at $x = \lambda_0$. Note that

$$\oint_{C(\lambda_0)} \frac{\partial}{\partial x} u dx = 0 \tag{1.6}$$

holds for any single-valued formal series u. Therefore, (A.2) is validated if we can find a single-valued formal series u for which the following relation holds:

$$\frac{\partial S_{\mathrm{odd}}}{\partial a} = \frac{\partial u}{\partial x}. \tag{A.3}$$

Another somewhat more sophisticated trial to confirm (A.2) was proposed by T. Aoki.

Proposition 1.2. *Let P denote the operator*

$$\frac{\partial^3}{\partial x^3} - 4\eta^2 Q_{\mathrm{II}} \frac{\partial}{\partial x} - 2\eta^2 \frac{\partial Q_{\mathrm{II}}}{\partial x} \tag{1.7}$$

and let ${}^tP$ denote its formal adjoint operator (, which actually coincides with $-P$). Then the following assertion (A.4) entails (A.3).
(A.4) *There exists a single-valued formal series w which satisfies*

$$\frac{\partial Q_{\mathrm{II}}}{\partial a} = {}^tPw. \tag{1.8}$$

To show this implication we need the following

Lemma 1.1. *(T. Aoki)*

$$\oint_{C(\lambda_0)} \frac{\partial S_{\mathrm{odd}}}{\partial a} dx = \oint_{C(\lambda_0)} \frac{\eta^2 \dfrac{\partial Q}{\partial a}}{2S_{\mathrm{odd}}} dx. \tag{1.9}$$

Proof. Although Aoki's original proof used the theory of variations, we present a more elementary and WKB-theoretic proof here. Let us first differentiate with respect to a the Riccati equation associated with (SL_{II}) :

$$2S\frac{\partial S}{\partial a} + \frac{\partial^2 S}{\partial a \partial x} = \eta^2 \frac{\partial Q}{\partial a}. \tag{1.10}$$

Comparison of even (with respect to the sign $\pm\sqrt{Q_0}$) parts in both sides of (1.10) gives us then

$$2\Big(S_{\text{odd}}\frac{\partial S_{\text{odd}}}{\partial a}+S_{\text{even}}\frac{\partial S_{\text{even}}}{\partial a}\Big)+\frac{\partial^2 S_{\text{even}}}{\partial a\partial x}=\eta^2\frac{\partial Q}{\partial a}. \tag{1.11}$$

Using the well-known relation

$$S_{\text{even}}=-\frac{\dfrac{\partial S_{\text{odd}}}{\partial x}}{2S_{\text{odd}}}, \tag{1.12}$$

we find

$$\frac{\eta^2\dfrac{\partial Q}{\partial a}}{2S_{\text{odd}}}-\frac{\partial S_{\text{odd}}}{\partial a}=\frac{1}{2}\frac{\partial}{\partial x}\Big(\frac{\dfrac{\partial S_{\text{even}}}{\partial a}}{S_{\text{odd}}}\Big). \tag{1.13}$$

This completes the proof of the lemma.

To prove Proposition 1.2 it suffices to note that P annihilates $\psi_+\psi_-$, where $\psi_\pm$ denotes the WKB solutions of (SL_{II}) given respectively in the form

$$\frac{1}{\sqrt{S_{\text{odd}}}}\exp\Big(\pm\int^x S_{\text{odd}}dx\Big). \tag{1.14}$$

Otherwise stated, $1/S_{\text{odd}}=\psi_+\psi_-$ is annihilated by the operator P. Hence the existence of a single-valued w satisfying (1.8) implies that the right-hand side of (1.9) vanishes. Thus (A.4) entails (A.2).

Our task is, thus, to find single-valued solutions of (A.3) or (1.8). This we will try in Section 2.

§2. Trials for constructing the required single-valued series.

To find single-valued solutions of (A.3) or (1.8), let us consider the following auxiliary equation:

$$\left(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} + A_x\right)v = A_a. \tag{2.1}$$

Here and in what follows, A_x (resp., A_a) denotes $\partial A/\partial x$ (resp., $\partial A/\partial a$). Note that $A = (2(x-\lambda))^{-1}$, as we are considering the case $J = \mathrm{II}$.

Lemma 2.1. *For any (i.e., not necessarily single-valued) solution v of (2.1), $u = vS_{\mathrm{odd}}$ and $w = v/(2\eta^2)$ satisfy the following relations:*

$$\frac{\partial S_{\mathrm{odd}}}{\partial a} = \frac{\partial u}{\partial x} + f \quad \textit{with} \quad \left(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} - A_x\right)f = 0, \tag{2.2}$$

$$\frac{\partial Q}{\partial a} = {}^tPw + g \quad \textit{with} \quad {}^tP = 4\eta^2 Q\frac{\partial}{\partial x} + 2\eta^2 Q_x - \frac{\partial^3}{\partial x^3} \tag{2.3}$$

$$\textit{and} \quad \left(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} - 2A_x\right)g = 0.$$

Proof. Differentiating the deformation equation (1.3) with respect to a, we find

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}A\right)\frac{\partial S_{\mathrm{odd}}}{\partial a} = \frac{\partial}{\partial x}(A_a S_{\mathrm{odd}}). \tag{2.4}$$

Hence we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}A\right)&\left(\frac{\partial S_{\mathrm{odd}}}{\partial a} - \frac{\partial u}{\partial x}\right)\\ &= \frac{\partial}{\partial x}(A_a S_{\mathrm{odd}}) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x}\right)u.\end{aligned} \tag{2.5}$$

On the other hand, (1.3) and (2.1) entail

$$\begin{aligned}\left(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x}\right)u &= (A_a + Av_x - A_x v)S_{\mathrm{odd}} + v\left(A_x S_{\mathrm{odd}} + A\frac{\partial S_{\mathrm{odd}}}{\partial x}\right)\\ &\quad - A\left(v_x S_{\mathrm{odd}} + v\frac{\partial S_{\mathrm{odd}}}{\partial x}\right) = A_a S_{\mathrm{odd}}.\end{aligned} \tag{2.6}$$

This implies that the right-hand side of (2.5) vanishes. Thus we have shown (2.2).

To show (2.3) we first recall that the compatibility of (SL_J) and its deformation equation (D_J) entails

$$\frac{\partial Q_J}{\partial t} = \frac{1}{2\eta^2}{}^tPA_J. \tag{2.7}$$

(See [AKT, §2].) Differentiating (2.7) with respect to a (with $J = \mathrm{II}$), we find the following relation (2.8) by the definition of ${}^tP$:

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} - 2A_x\Big)Q_a = \frac{1}{2\eta^2}{}^tPA_a. \tag{2.8}$$

On the other hand, a direct computation shows

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} - 2A_x\Big){}^tP = {}^tP\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} + A_x\Big). \tag{2.9}$$

Hence (2.1) together with (2.8) and (2.9) implies the following:

$$\begin{aligned}&\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} - 2A_x\Big)(Q_a - {}^tPw)\\ &= \frac{1}{2\eta^2}{}^tPA_a - {}^tP\Big(\frac{A_a}{2\eta^2}\Big) = 0.\end{aligned} \tag{2.10}$$

This shows (2.3). Q.E.D.

Now our task is to construct a single-valued solution v of (2.1) so that f or g may vanish. Before reporting our trials, let us note the constraints on f and g are not so stingy as one might imagine. In fact we have the following

Proposition 2.1. *Let n be an integer and let $f = \sum_{j\geq 0} f_{j/2}\eta^{-j/2}$ be a single-valued series that satisfies*

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} - nA_x\Big)f = 0. \tag{2.11}$$

Assume that the free parameters α_0 and β_0 that determine $\lambda_{\mathrm{II}}(t, a, \eta; \alpha_0, \beta_0)$ are both different from 0. Assume further that $f_{j/2}$ consists of k-instanton *terms with k odd and $|k| \le j$ if j is odd, and k even and $|k| \le j$ if j is even. Then there exists a formal series $c = \sum_{j\ge 0} c_j \eta^{-j}$ that satisfies the following:*

$$\frac{\partial c_j}{\partial t} = \frac{\partial c_j}{\partial x} = 0 \quad \text{for any } j. \tag{2.12}$$

$$f = c(\eta^{-1} S_{\text{odd}})^n. \tag{2.13}$$

Proof. Let us first note that $u = f(\eta^{-1} S_{\text{odd}})^{-n}$ satisfies

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x}\Big)u = 0. \tag{2.14}$$

In fact, a straightforward computation using (1.3) shows

$$\begin{aligned} \eta^{-n}\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x}\Big)u & \\ = nA_x f(S_{\text{odd}})^{-n} &+ f(-n(S_{\text{odd}})^{-n-1}(A_x S_{\text{odd}} + AS_{\text{odd},x}) \\ &+ nA(S_{\text{odd}})^{-n-1} S_{\text{odd},x}) = 0. \end{aligned} \tag{2.15}$$

Clearly $u = \sum_{j\ge 0} u_{j/2}\eta^{-j/2}$ is single-valued. Furthermore its instanton structure is the same as that of f, because $S_{\text{odd}} - \eta S_{-1}$ has the same instanton structure as that of f and because S_{-1} is free from instanton terms. Hence it suffices for us to verify that such a solution u of (2.14) is actually a constant series in the sense of (2.12).

In our argument the fact that the differentiation of an instanton term with respect to t enhances the degree of the term in η plays an important role. Hence we introduce the following notations:

(2.16) $\overset{\triangle}{\frac{\partial}{\partial t}} u_{j/2}$ denotes the part of $\dfrac{\partial u_{j/2}}{\partial t}$ which contains an extra-factor η (through the differentiation of $\exp(\pm \ell\phi(t,a)\eta)$).

(2.17) $\overset{\circ}{\frac{\partial}{\partial t}} u_{j/2}$ is, by definition, $\dfrac{\partial u_{j/2}}{\partial t} - \overset{\triangle}{\frac{\partial}{\partial t}} u_{j/2}$.

We now expand A in $\eta^{-1/2}$ using the expansion of λ_{II} and equate the coefficients of like powers of η in (2.14): the highest degree term in (2.14), i.e., $(\underset{\triangle}{\frac{\partial}{\partial t}} u_{1/2})\eta^{-1/2}$(, which is actually of degree $\pm 1/2$ not $-1/2$,) should vanish; since $u_{1/2}$ consists of (± 1)-instanton terms, this means $u_{1/2}$ should vanish. ¿From the coefficient of η^0 we next find:

(2.18) $$\Big(\frac{\partial}{\partial t} - A_0 \frac{\partial}{\partial x}\Big) u_0 + \underset{\triangle}{\frac{\partial}{\partial t}} u_1 = 0.$$

Since A_0 and u_0 consist of 0-instanton terms and since $\underset{\triangle}{\frac{\partial}{\partial t}} u_1$ does not contain 0-instanton terms by the definition, (2.18) entails

(2.19) $$\Big(\frac{\partial}{\partial t} - A_0 \frac{\partial}{\partial x}\Big) u_0 = 0$$

and that

(2.20) (± 2)-instanton terms of u_1 should vanish.

The relation

(2.21) $$\underset{\triangle}{\frac{\partial}{\partial t}} u_{3/2} - A_{1/2} \frac{\partial}{\partial x} u_0 = 0,$$

which results from the comparison of the coefficients of $\eta^{-1/2}$, will play an important role at the next stage (i.e., in proving the vanishing of $u_{3/2}$), but it does not help us at this stage. The comparison of the coefficients of η^{-1} gives

(2.22) $$\underset{\triangle}{\frac{\partial}{\partial t}} u_2 + \underset{\bigcirc}{\frac{\partial}{\partial t}} u_1 - \Big(A_1 \frac{\partial u_0}{\partial x} + A_0 \frac{\partial u_1}{\partial x}\Big) = 0.$$

Since we have confirmed that (± 2)-instanton terms of u_1 vanish, we find

(2.23) $$\underset{\bigcirc}{\frac{\partial}{\partial t}} u_1 = \frac{\partial}{\partial t} u_1.$$

On the other hand, using the explicit computation of λ_{II} (cf. [AKT, Appendix]), we find that the 0-instanton term of A_1 is given by the following:

(2.24) $$\frac{\lambda_0 \Delta^{-3/2}(-6\alpha_0\beta_0)}{(x-\lambda_0)^2} + \frac{\Delta^{-1/2}\alpha_0\beta_0}{(x-\lambda_0)^3},$$

where $\Delta = (6\lambda_0^2 + t)$. Hence the comparison of 0-instanton terms in (2.22) entails

$$(2.25)\quad \left(\frac{\partial}{\partial t} - A_0 \frac{\partial}{\partial x}\right) u_1 = \left(\frac{-6\alpha_0\beta_0\lambda_0\Delta^{-3/2}}{(x-\lambda_0)^2} + \frac{\alpha_0\beta_0\Delta^{-1/2}}{(x-\lambda_0)^3}\right)\frac{\partial u_0}{\partial x}.$$

We now expand u_0 as $\sum_{j\geq -N} c_j(t)(x-\lambda_0)^j$ and substitute it into (2.19). Since $A_0 = 1/(2(x-\lambda_0))$, we immediate find $N = 0$. We further find

(2.26) $c_1(t) = 0,$

(2.27) $c'_k(t) - (k+1)c_{k+1}(t)\lambda'_0(t) + (k+2)c_{k+2}(t)/2 = 0 \quad (k \geq 0),$

while no condition is imposed upon c_0 at this stage. Let us also expand u_1 as $\sum_{j\geq M} d_j(x-\lambda_0)^j$ and substitute it into (2.25). As we find by (2.26) that the right-hand side of (2.25) begins with $2c_2\alpha_0\beta_0\Delta^{-1/2}/(x-\lambda_0)^2$, M should be 0. But then the left-hand side of (2.25) begins with $(-d_1/2(x-\lambda_0))$, lacking a double-pole term. Hence we conclude

$$(2.28)\qquad c_2\alpha_0\beta_0\Delta^{-1/2} = 0.$$

Since $\alpha_0\beta_0 \neq 0$ by the assumption, (2.27) with $k = 0$ then implies

$$(2.29)\qquad c'_0 = 0.$$

Hence it follows from (2.27) that c_k vanishes for all $k \geq 1$. Therefore u_0 is a constant independent of both x and t. This fact also implies that (2.21) now takes the following form:

$$(2.21')\qquad \frac{\partial}{\partial t} u_{3/2} = 0.$$

Since $u_{3/2}$ does not contain a 0-instanton term by the assumption, (2.21′) implies the vanishing of $u_{3/2}$, in just the same way as in the verification of the vanishing of $u_{1/2}$. We also note the vanishing of $\partial u_0/\partial x$ renders (2.22) the same as (2.18) with the indices of u shifted. Thus our reasoning goes on exactly in the same way as before, and we find u_1 is also independent of both x and t. Repeating this, we conclude

$$(2.30)\qquad u = \sum_{j\geq 0} c_j \eta^{-j}$$

with

$$\frac{\partial c_j}{\partial x} = \frac{\partial c_j}{\partial t} = 0 \quad \text{for any } j. \tag{2.31}$$

Q.E.D.

This proposition together with Lemma 2.1 might make the reader suspect that a suitable choice of a solution v of (2.1) would kill f or g, as the freedom of f and g is relatively small, i.e., just a constant series as shown above. Unfortunately, Proposition 2.1 itself nullifies such a hope; the arbitrariness of v is given by the addition of h that satisfies

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} + A_x\Big)h = 0. \tag{2.32}$$

Proposition 2.1 applies also to (2.32), and in our current situation we find

$$h = \tilde{c}(S_{\mathrm{odd}})^{-1}\eta \tag{2.33}$$

with some constant series $\tilde{c}$. Then it is clear that $\partial u/\partial x$ is kept intact under the addition of h to v in Lemma 2.1. Concerning (2.3), let us recall that ${}^tP$ annihilates $(S_{\mathrm{odd}})^{-1}$. Hence ${}^tPw$ is also kept intact by the addition of $h/2\eta^2$. Thus there remains a missing link between the construction of v to be given below and the vanishing of f or g, although we believe it to be true. Because of this trouble this report is still incomplete. Hence here we content ourselves with describing how to construct a single-valued solution v of (2.1). In what follows we always consider the problem near $x = \lambda_0(t)$ (or, to be more precise, on a fixed neighborhood of $x = \lambda_0(t)$) and we usually do not mention it.

Let us start with a holomorphic solution $w_0(x,t)$ of the following equation:

$$\Big(2(x-\lambda_0)^2\frac{\partial}{\partial t} - (x-\lambda_0)\frac{\partial}{\partial x} - 1\Big)w_0 = \frac{\partial \lambda_0}{\partial a}. \tag{2.34}$$

It is obvious that this equation corresponds to the top degree part of (2.1) as far as non-instanton terms are concerned. The unique existence of holomorphic solution can be readily confirmed if we seek for w_0 by expanding it as $\sum_{j\geq 0} c_j(t)(x-\lambda_0(t))^j$; note that the crucial index in this construction is $j = -1$ in view of the explicit form of (2.34). The fact that the crucial index is $j = -1$ also implies the existence of an analytic solution $\chi_{-1/2} = \sum_{j \geq -1} \tilde{c}_j(t)(x - \lambda_0(t))^j$ of the following equation:

$$(2.35) \qquad (2(x-\lambda_0)^2 \frac{\partial}{\partial t} - (x-\lambda_0)\frac{\partial}{\partial x} - 1)\chi_{-1/2} = 0.$$

One important point to be noted here is that $\tilde{c}_{-1}(t)$ can be an arbitrary analytic function; we later use this freedom to construct the required single-valued solution.

To proceed further we note that the differentiation of λ with respect to a enhances the degree of η when it is applied to instanton terms, just in the same manner as in the case of differentiation with respect to t. Hence the highest degree term in η in the right-hand side of (2.1) is not $\lambda_{0,a}/(2(x-\lambda_0)^2)$ but $(\eta^{-1/2}\frac{\partial}{\partial a}\Lambda_0)/(2(x-\lambda_0)^2)$, when λ is expanded as

$$(2.36) \qquad \lambda = \lambda_0 + \eta^{-1/2}(\Lambda_0 + \eta^{-1/2}\Lambda_{1/2} + \cdots).$$

In order to compensate this term, we seek for the solution v with the following expansion:

$$(2.37) \quad v = (w_0 + \chi_{-1/2}) + \eta^{-1/2}(V_0 + \eta^{-1/2}V_{1/2} + \eta^{-1}V_1 + \cdots),$$

where $V_{2j/2}$ consists of $\pm(2k+1)$-instanton terms with $0 \leq k \leq j$ and $V_{(2j+1)/2}$ consists of $\pm 2k$-instanton terms with $0 \leq k \leq j+1$. As A_a contains arbitrarily high-order poles at $x = \lambda_0$, we are to be prepared for pole singularities in $V_{j/2}$. To cope with instanton terms neatly, we introduce a new variable τ, which is later set to be $\eta\phi$, and we employ the multiple-scale analysis to find v of the form (2.37) so that it may satisfy the following equation.

(2.38)

$$(2(x-\lambda)^2\Big(\eta^{-1}\frac{\partial}{\partial t} + \phi'\frac{\partial}{\partial \tau}\Big) - \eta^{-1}(x-\lambda)\frac{\partial}{\partial x} - \eta^{-1})(\eta^{1/2}(w_0 + \chi_{-1/2}) + V)$$
$$= \eta^{-1/2}\lambda_a.$$

Here and in what follows, ϕ' denotes $\partial\phi(t,a)/\partial t$. Since (2.38) is a linear differential equation, the application of multiple-scale analysis is straightforward. However, we should also keep in mind the extra-requirement that V should be single-valued. In order to be more explicit, let $g = g_0 + \eta^{-1/2} g_{1/2} + \eta^{-1} g_1 + \cdots$ denote $\eta^{-1/2}(\lambda_a - \lambda_{0,a})$. Then, by equating the coefficients of like powers of η in (2.38), we find

$$2(x-\lambda_0)^2 \phi' \frac{\partial}{\partial \tau} V_0 = g_0, \tag{2.39}$$

$$2(x-\lambda_0)^2 \phi' \frac{\partial}{\partial \tau} V_{1/2} = 4(x-\lambda_0)\Lambda_0 \phi' \frac{\partial}{\partial \tau} V_0 + g_{1/2}, \tag{2.40}$$

and so on. As we have required that V_0 consists of (± 1)-instanton terms, (2.39) can be algebraically solved. Equation (2.40), however, determines only (± 2)-instanton terms of $V_{1/2}$, leaving its 0-instanton term undetermined. Let this undetermined term be denoted by $\varphi_{1/2}$. Here we note two peculiar features of (2.40); first, $g_{1/2}$ does not contain a 0-instanton term by its definition, and, second, $\Lambda_0(\partial V_0/\partial \tau)$ does not contain a 0-instanton term either, by the explicit form of g_0 and V_0. These two facts combined enable us to find the (± 2)-instanton part of $V_{1/2}$ in an algebraic manner with its 0-instanton part undetermined; if there were some 0-instanton term in the right-hand side of (2.40), $V_{1/2}$ should contain a linear term in τ. As τ should be eventually set to be $\eta\phi$, the appearance of such a term should jeopardize the filtration of v assumed in (2.37). Note that the function $\chi_{-1/2}$ is irrelevant (and hence cannot help us) at this level. Actually $\chi_{-1/2}$ becomes relevant later in constructing a single-valued $\varphi_{1/2}$.

Now, looking at the coefficients of $\eta^{-1/2}$ and η^{-1} in (2.38), we find the equations:

$$\begin{aligned} &2(x-\lambda_0)^2 \phi' \frac{\partial}{\partial \tau} V_1 - 4\Lambda_0 (x-\lambda_0)\phi' \frac{\partial}{\partial \tau} V_{1/2} \\ &+ 2(\Lambda_0^2 - 2(x-\lambda_0)\Lambda_{1/2})\phi' \frac{\partial}{\partial \tau} V_0 \\ &+ \big(2(x-\lambda_0)^2 \frac{\partial}{\partial t} - (x-\lambda_0)\frac{\partial}{\partial x} - 1\big) V_0 \\ &= g_1 + \big(4(x-\lambda_0)\Lambda_0 \frac{\partial}{\partial t} - \Lambda_0 \frac{\partial}{\partial x}\big)(v_0 + \chi_{-1/2}), \end{aligned} \tag{2.41}$$

$$\begin{aligned} &2(x-\lambda_0)^2 \phi' \frac{\partial}{\partial \tau} V_{3/2} - 4(x-\lambda_0)\Lambda_0 \phi' \frac{\partial}{\partial \tau} V_1 \\ &+ 2(\Lambda_0^2 - 2(x-\lambda_0)\Lambda_{1/2})\phi' \frac{\partial}{\partial \tau} V_{1/2} \end{aligned} \tag{2.42}$$

$$
\begin{aligned}
&+ 2(2\Lambda_0\Lambda_{1/2} - 2(x-\lambda_0)\Lambda_1)\phi'\frac{\partial}{\partial\tau}V_0 \\
&+ \Big(2(x-\lambda_0)^2\frac{\partial}{\partial t} - (x-\lambda_0)\frac{\partial}{\partial x} - 1\Big)V_{1/2} \\
&+ \Big(-4(x-\lambda_0)\Lambda_0\frac{\partial}{\partial t} + \Lambda_0\frac{\partial}{\partial x}\Big)V_0 \\
&= g_{3/2} + \Big(4(x-\lambda_0)\Lambda_{1/2}\frac{\partial}{\partial t} - 2\Lambda_0^2\frac{\partial}{\partial t} - \Lambda_{1/2}\frac{\partial}{\partial x}\Big)(v_0 + \chi_{-1/2}).
\end{aligned}
$$

Since V_1 is supposed to consist of only odd instanton terms, (2.41) determines V_1 in an algebraic manner. At the level of (2.42), we encounter the non-secularity condition: To find $V_{3/2}$ without a term linearly dependent on τ, we require the 0-instanton term $\varphi_{1/2}$ contained in $V_{1/2}$ should be chosen so that the following holds:

(2.43) $\Big(2(x-\lambda_0)^2\frac{\partial}{\partial t} - (x-\lambda_0)\frac{\partial}{\partial x} - 1\Big)\varphi_{1/2}$ cancels out the sum of all other 0-instanton terms in (2.42).

A lengthy computation using the results in Appendix of [AKT] shows that the explicit form of the requirement (2.43) is as follows:

$$
\Big(2(x-\lambda_0)^2\frac{\partial}{\partial t} - (x-\lambda_0)\frac{\partial}{\partial x} - 1\Big)\varphi_{1/2} = \sum_{j\geq -3} f_j(x-\lambda_0)^j, \tag{2.44}
$$

with

$$
f_{-3} = 4a_1^{(0)}a_{-1}^{(0)}\tilde{c}_{-1}, \tag{2.45}
$$

$$
f_{-2} = 12a_1^{(0)}a_{-1}^{(0)}\frac{\partial\lambda_0}{\partial t}\tilde{c}_{-1} + a_0^{(1/2)}\tilde{c}_{-1}, \tag{2.46}
$$

$$
f_{-1} = 2\frac{\partial}{\partial a}(a_1^{(0)}a_{-1}^{(0)}) + 4a_0^{(1/2)}\frac{\partial\lambda_0}{\partial t}\tilde{c}_{-1} + 8a_1^{(0)}a_{-1}^{(0)}\frac{\partial\tilde{c}_{-1}}{\partial t}, \tag{2.47}
$$

where $a_k^{(j)}$ denotes the coefficient of $\exp(k\phi\eta)$ in Λ_j. If we expand $\varphi_{1/2}$ as $\sum_{j\geq -3} d_j(t)(x-\lambda_0(t))^j$ and substitute it into (2.44), we find

$$
2d_{-3} = f_{-3}, \tag{2.48}
$$

$$
6d_{-3}\frac{\partial\lambda_0}{\partial t} + d_{-2} = f_{-2}, \tag{2.49}
$$

$$
2\frac{\partial d_{-3}}{\partial t} + 4d_{-2}\frac{\partial\lambda_0}{\partial t} + 0\cdot d_{-1} = f_{-1}, \tag{2.50}
$$

and other d_j $(j \geq 0)$ is determined recursively by d_k and $f_{k'}$ $(k < j,\ k' \leq j)$ on the condition that d_j $(j \leq 0)$ is found. The fact that the coefficient of d_{-1} in (2.50) is 0 implies the following two facts:

(2.51) there exists a non-trivial relation among f_j's $(-3 \leq j \leq -1)$,

(2.52) if $\varphi_{1/2}$ solves (2.44), then $\varphi_{1/2} + \chi_{1/2}$ also solves (2.44), where $\chi_{1/2}$ has the form

(2.53) $\sum_{j \geq -1} \tilde{d}_j(t)(x - \lambda_0)^j$

and it satisfies (2.44) with all f_j vanishing, namely,

$$(2(x-\lambda_0)^2 \frac{\partial}{\partial t} - (x-\lambda_0)\frac{\partial}{\partial x} - 1)\chi_{1/2} = 0. \tag{2.54}$$

Here we note that $\tilde{d}_{-1}(t)$ can be arbitrarily chosen. The relation among f_j's $(-3 \leq j \leq -1)$ results from (2.48) $\sim$ (2.50), and its explicit form is as follows:

$$f_{-1} = \frac{\partial f_{-3}}{\partial t} + 4\frac{\partial \lambda_0}{\partial t}(f_{-2} - 3\frac{\partial \lambda_0}{\partial t} f_{-3}). \tag{2.55}$$

Substitution of (2.45) $\sim$ (2.47) into (2.55) then entails

$$\begin{aligned} &2\frac{\partial}{\partial a}(a_1^{(0)} a_{-1}^{(0)}) + 4a_0^{(1/2)}\frac{\partial \lambda_0}{\partial t}\tilde{c}_{-1} + 8a_1^{(0)} a_{-1}^{(0)}\frac{\partial \tilde{c}_{-1}}{\partial t} \\ &= \frac{\partial}{\partial t}(4a_1^{(0)} a_{-1}^{(0)}\tilde{c}_{-1}) + 4\frac{\partial \lambda_0}{\partial t}a_0^{(1/2)}\tilde{c}_{-1}, \end{aligned} \tag{2.56}$$

or equivalently,

$$2\frac{\partial}{\partial a}(a_1^{(0)} a_{-1}^{(0)}) + 4a_1^{(0)} a_{-1}^{(0)}\frac{\partial \tilde{c}_{-1}}{\partial t} - 4\frac{\partial}{\partial t}(a_1^{(0)} a_{-1}^{(0)})\tilde{c}_{-1} = 0. \tag{2.57}$$

Since

$$a_1^{(0)} a_{-1}^{(0)} = \alpha_0 \beta_0 (6\lambda_0^2 + t)^{-1/2} \tag{2.58}$$

holds (cf. [AKT, Appendix]), we can find a multi-valued analytic solution $\tilde{c}_{-1}$ of the equation (2.57). Note that the single-valuedness we are concerned with is with respect to $x - \lambda_0$, not with respect to t.

Let us now discuss how the function $w_0 + \chi_{-1/2}$ thus constructed is related to the relation (2.2). First let us expand S_{-1} as follows:

$$
\begin{aligned}
(2.59) \qquad S_{-1} = \sqrt{\Delta}(x-\lambda_0) &+ \frac{2\lambda_0}{\sqrt{\Delta}}(x-\lambda_0)^2 \\
&+ \Big(\frac{1}{2\sqrt{\Delta}} - \frac{2\lambda_0^2}{\Delta^{3/2}}\Big)(x-\lambda_0)^3 + \cdots
\end{aligned}
$$

with $\Delta = 6\lambda_0^2 + t$. We also find

$$
\begin{aligned}
(2.60) \qquad w_0 + \chi_{-1/2} = \tilde{c}_{-1}(x-\lambda_0)^{-1} &+ \Big(2\tilde{c}_{-1}\frac{\partial\lambda_0}{\partial t} - \frac{\partial\lambda_0}{\partial a}\Big) \\
&+ \frac{\partial\tilde{c}_{-1}}{\partial t}(x-\lambda_0) + \cdots .
\end{aligned}
$$

Hence we obtain the following:

$$
(2.61) \qquad \frac{\partial S_{-1}}{\partial a} = -\sqrt{\Delta}\frac{\partial\lambda_0}{\partial a} + \Big(\frac{\partial\sqrt{\Delta}}{\partial a} - \frac{4\lambda_0\lambda_{0,a}}{\sqrt{\Delta}}\Big)(x-\lambda_0) + \cdots,
$$

$$
\begin{aligned}
(2.62) \qquad \frac{\partial}{\partial x}((w_0 + \chi_{-1/2})S_{-1}) &= (2\tilde{c}_{-1}\frac{\partial\lambda_0}{\partial t} - \frac{\partial\lambda_0}{\partial a})\sqrt{\Delta} + \frac{2\tilde{c}_{-1}\lambda_0}{\sqrt{\Delta}} \\
&+ 2\Big\{\frac{\partial\tilde{c}_{-1}}{\partial t}\sqrt{\Delta} + \frac{2\lambda_0}{\sqrt{\Delta}}(2\tilde{c}_{-1}\frac{\partial\lambda_0}{\partial t} - \frac{\partial\lambda_0}{\partial a}) \\
&+ \tilde{c}_{-1}(\frac{1}{2\sqrt{\Delta}} - \frac{2\lambda_0^2}{\Delta^{3/2}})\Big\}(x-\lambda_0) + \cdots .
\end{aligned}
$$

On the other hand, it follows from the definition of λ_0, i.e., $2\lambda_0^3 + t\lambda_0 + a = 0$, that

$$
(2.63) \qquad \Delta\frac{\partial\lambda_0}{\partial t} + \lambda_0 = 0.
$$

Hence the constant term of (2.61) is identical with that of (2.62). Using (2.63) again, we deduce the following relation (2.64) from the coincidence of the coefficient of $(x-\lambda_0)$ in (2.61) and that in (2.62):

$$
(2.64) \qquad \frac{1}{2}\Delta^{-1/2}\Delta_a = 2\sqrt{\Delta}\frac{\partial\tilde{c}_{-1}}{\partial t} + \Big(\frac{1}{\sqrt{\Delta}} - \frac{12\lambda_0^2}{\Delta^{3/2}}\Big)\tilde{c}_{-1}.
$$

One can readily verify that (2.64) is equivalent to (2.57) if $\alpha_0\beta_0 \neq 0$. Thus we have verified that f_{-1}, the coefficient of η^1 of f in (2.2), is of order $(x-\lambda_0)^2$. Proposition 2.1 implies that $f_{-1} = c_0 S_{-1}$ holds for some constant c_0. Since S_{-1} has simple zero at $x = \lambda_0$, this means $c_0 = 0$. Hence the top term of our solution, i.e., $w_0 + \chi_{-1/2}$, satisfies the required condition. Note also that each solution $\tilde{c}_{-1}(t)$ of (2.64) (or, equivalently (2.57)) has the following form:

$$\tilde{c}_{-1} = \frac{\phi_a + \gamma_{-1}}{2\sqrt{\Delta}} \quad \text{with a complex number } \gamma_{-1}. \tag{2.65}$$

The concrete form (2.59) of S_{-1} indicates that the arbitrary constant γ_{-1} corresponds to the top term of the arbitrary function h given by (2.33), while the substantial part $\phi_a/(2\sqrt{\Delta})$ coincides with

$$\frac{\partial}{\partial a}\Big(\int_s^{\lambda_0} S_{-1}dx\Big)/S_{-1}, \tag{2.66}$$

where s denotes a simple turning point of (SL_{II}). Here we have used the fact

$$\int_s^{\lambda_0} S_{-1}dx = \frac{1}{2}\phi. \tag{2.67}$$

(See [KT1, Proposition 2.1].) This fact seems to be worth mentioning in connection with Remark 3.2 to be given later.

The way how to construct a single-valued v satisfying (2.38) is now evident; the non-secularity condition can be described in terms of linear differential equation for $\varphi_{(2j+1)/2}$, and, to find a single-valued $\varphi_{(2j+1)/2}$, we should choose an appropriate $\chi_{(2j-1)/2}$, a null solution of the equation for $\varphi_{(2j-1)/2}$. Our hope is that the solution v of (2.38) thus constructed should satisfy the relation

$$\frac{\partial S_{\mathrm{odd}}}{\partial a} = \frac{\partial}{\partial x}(v S_{\mathrm{odd}}). \tag{2.68}$$

So far, however, we have been unable to confirm this.

§3. Can we deform (SL_{II}) in a-variable?

The principal aim of this section is to show that Ansatz 3.1 or Ansatz 3.2 below lets (SL_{II}) be deformed in a-variable.

Ansatz 3.1. There exists a single-valued solution v of (2.1) so that it satisfies

$$\frac{\partial S_{\text{odd}}}{\partial a} = \frac{\partial}{\partial x}(vS_{\text{odd}}). \tag{3.1}$$

Ansatz 3.2. There exists a single-valued solution v of (2.1) so that it satisfies

$$2\eta^2\frac{\partial Q}{\partial a} = {}^tPv\Big(\underset{\text{by def.}}{=} 4\eta^2 Q\frac{\partial v}{\partial x} + 2\eta^2\frac{\partial Q}{\partial x}v - \frac{\partial^3 v}{\partial x^3}\Big). \tag{3.2}$$

Remark 3.1. If $\alpha_0\beta_0 \neq 0$ and if v is single-valued, then f and g in Lemma 2.1 are related in the following manner:

$$f^2 = \frac{c\eta^2 g}{2} \tag{3.3}$$

with c being a constant series.

In fact, a straightforward computation shows

$$\begin{aligned}&\frac{\partial(S_{\text{odd}} + S_{\text{even}})}{\partial a} - (vS_{\text{odd}})_x - (vS_{\text{even}})_x \\ &= f + \frac{1}{2{S_{\text{odd}}}^2}(S_{\text{odd},x}f - S_{\text{odd}}f_x) - \frac{1}{2}v_{xx}.\end{aligned} \tag{3.4}$$

On the other hand, Proposition 2.1 asserts that $f = cS_{\text{odd}}$ holds for some constant series c. Thus we obtain

$$\frac{\partial S}{\partial a} = \frac{\partial}{\partial x}(vS) - \frac{1}{2}v_{xx} + cS_{\text{odd}}. \tag{3.5}$$

Differentiating with respect to a the Riccati equation that S satisfies, we also find

$$\eta^2 Q_a = 2SS_a + S_{xa}. \tag{3.6}$$

Substitution of (3.5) into (3.6) then entails

$$\begin{aligned}
\eta^2 Q_a &= 2S(Sv_x + S_x v - \frac{1}{2}v_{xx}) + 2Sf \\
&\quad + (S_{xx}v + 2S_x v_x + Sv_{xx}) - \frac{1}{2}v_{xxx} + f_x \\
&= (2S^2 + 2S_x)v_x + (S^2 + S_x)_x v - \frac{1}{2}v_{xxx} + 2Sf + f_x \\
&= 2\eta^2 Q v_x + \eta^2 Q_x v - \frac{1}{2}v_{xxx} + 2Sf + f_x \\
&= \eta^2 Q_a - \eta^2 g + 2Sf + f_x.
\end{aligned} \tag{3.7}$$

Hence we find

$$\eta^2 g = 2Sf + f_x. \tag{3.8}$$

Substitution of $f = cS_{\mathrm{odd}}$ into (3.8) leads to the following:

$$\begin{aligned}
\eta^2 g &= 2\left(S_{\mathrm{odd}} - \frac{S_{\mathrm{odd},x}}{2S_{\mathrm{odd}}}\right)cS_{\mathrm{odd}} + cS_{\mathrm{odd},x} \\
&= 2c{S_{\mathrm{odd}}}^2 = \frac{2f^2}{c}.
\end{aligned} \tag{3.9}$$

This proves (3.3). Note also that the above reasoning (in particular, (3.7)) shows that Ansatz 3.1 automatically entails Ansatz 3.2.

Remark 3.2. As the preceding remark implies, Ansatz 3.1 and Ansatz 3.2 are equivalent if $\alpha_0\beta_0 \neq 0$. On the other hand, if $\alpha_0\beta_0 = 0$ Ansatz 3.1 (and hence Ansatz 3.2 also) can be validated as follows; when $\alpha_0\beta_0 = 0$, we can show E vanishes for pure solutions $\lambda_{\mathrm{II}}(t, a, \eta; \alpha_0, \beta_0)$. Since we know

$$E = \frac{2}{\pi i}\oint_{C(\lambda_0)} S_{\mathrm{odd}}\,dx, \tag{3.10}$$

the following function $\mathcal{V}$ is single-valued:

$$\mathcal{V} = \Big(\frac{\partial}{\partial a}\int_s^x S_{\mathrm{odd}}dx\Big)/(S_{\mathrm{odd}}), \tag{3.11}$$

where s is a simple turning point of (SL_{II}). On the other hand, it follows from the deformation (in t) equation (1.3) for S_{odd} that

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x}\Big)\Big(\int_s^x S_{\mathrm{odd}}dx\Big) = 0, \tag{3.12}$$

and hence we find

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial a}\int_s^x S_{\mathrm{odd}}dx\Big) = A_a S_{\mathrm{odd}}. \tag{3.13}$$

Therefore $\mathcal{V}$ satisfies (2.1), that is,

$$\Big(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x} + A_x\Big)\mathcal{V} = A_a. \tag{3.14}$$

Furthermore it is trivial that

$$\frac{\partial S_{\mathrm{odd}}}{\partial a} = \frac{\partial}{\partial x}(\mathcal{V} S_{\mathrm{odd}}) \tag{3.15}$$

holds. This means f in (2.2) actually vanishes for the single-valued solution $\mathcal{V}$ of (2.1). Of course this observation does not bear any importance in making the assertion (A.2); E vanishes identically in this situation, and so it is trivially independent of a. However, this fact is important in the discussion below, as this function $\mathcal{V}$ guarantees the deformation of (SL_{II}) in a.

Proposition 3.1. *Suppose that Ansatz 3.1 or Ansatz 3.2 is validated. Let L, M and N respectively denote the following operators:*

$$L = \frac{\partial^2}{\partial x^2} - \eta^2 Q, \tag{3.16}$$

$$M = \frac{\partial}{\partial t} - A\frac{\partial}{\partial x} + \frac{1}{2}A_x, \tag{3.17}$$

$$N = \frac{\partial}{\partial a} - v\frac{\partial}{\partial x} + \frac{1}{2}v_x. \tag{3.18}$$

Then the following commutation relations hold:

$$[L, M] = -2A_x L, \tag{3.19}$$
$$[L, N] = -2v_x L, \tag{3.20}$$
$$[M, N] = 0. \tag{3.21}$$

Proof. The relation (3.19) is well-known (, and actually its proof is essentially the same as that given below to show (3.20)). As noted in Remark 3.1, Ansatz 3.1 automatically entails Ansatz 3.2. Hence we suppose that Ansatz 3.2 is validated. A direct computation shows

$$\begin{aligned}[L, N] &= -2v_x \frac{\partial^2}{\partial x^2} + \frac{v_{xxx}}{2} + \eta^2 Q_a - \eta^2 v \frac{\partial Q}{\partial x} \\ &= -2v_x L + \eta^2 Q_a - \frac{{}^tPv}{2},\end{aligned} \tag{3.22}$$

and hence Ansatz 3.2 implies (3.20). The verification of (3.21) is based only on the fact that v satisfies the relation (2.1) (rather than on the whole Ansatz);

$$\begin{aligned}[M, N] &= (-v_t + Av_x + A_a - vA_x)\frac{\partial}{\partial x} \\ &\quad + \frac{1}{2}(v_t - Av_x - A_a + vA_x)_x \\ &= 0\end{aligned} \tag{3.23}$$

by (2.1).

Q.E.D.

Remark 3.3. Proposition 3.1 indicates a symmetry between t and a. To emphasize this aspect of the problem, it might be interesting to rewrite our starting relation (2.1) and our goal (3.1) respectively by the following ones:

$$\left(\frac{\partial}{\partial t} - A\frac{\partial}{\partial x}\right)v = \left(\frac{\partial}{\partial a} - v\frac{\partial}{\partial x}\right)A, \tag{2.1$'$}$$
$$\frac{\partial}{\partial t}(vS_{\mathrm{odd}}) = \frac{\partial}{\partial a}(AS_{\mathrm{odd}}). \tag{3.1$'$}$$

Note that (3.1′) and (3.1) are equivalent under (2.1). Similarly the resemblance between (2.7) and (3.2) should be observed.

Remark 3.4. If we start our discussion with (2.3) (instead of (3.2)), then we find the following (3.20′) instead of (3.20):

$$[L, N] = -2v_x L + \eta^2 g. \tag{3.20′}$$

Furthermore the constraint on g given in (2.3) is then a consequence of the Jacobi identity for L, M and N, namely $[L,[M,N]] + [M,[N,L]] + [N,[L,M]] = 0$.

Now Proposition 3.1 asserts that the following system $\mathcal{N}$ of differential equations is in involution if we suppose Ansatz 3.1 or Ansatz 3.2:

$$\mathcal{N} : L\psi = M\psi = N\psi = 0. \tag{3.24}$$

In view of the close resemblance bteween $\mathcal{N}$ and the hitherto known couple of differential equations $L\psi = M\psi = 0$, we can readily imagine that WKB analysis of (3.24) may be possible. In fact, we find the following

Proposition 3.2. *Suppose that Ansatz 3.1 is validated. Then the following* 1*-form* ω *is closed:*

$$\omega = Sdx + (AS - \frac{1}{2}A_x)dt + (vS - \frac{1}{2}v_x)da. \tag{3.25}$$

Proof. Let us first show (3.1) entails

$$\frac{\partial S}{\partial a} = \frac{\partial}{\partial x}(vS) - \frac{1}{2}v_{xx}. \tag{3.26}$$

In fact, using $S_{\text{even}} = -S_{\text{odd},x}/(2S_{\text{odd}})$, we find (3.27) below by (3.1):

$$\begin{aligned} &\frac{\partial S}{\partial a} - \frac{\partial}{\partial x}(vS) + \frac{1}{2}v_{xx} \\ &= \frac{S_{\text{odd},x}(vS_{\text{odd}})_x}{2S_{\text{odd}}{}^2} - \frac{(vS_{\text{odd}})_{xx}}{2S_{\text{odd}}} \\ &\quad - \frac{vS_{\text{odd},x}{}^2}{2S_{\text{odd}}{}^2} + \frac{(vS_{\text{odd},x})_x}{2S_{\text{odd}}} + \frac{1}{2}v_{xx} \\ &= \frac{v_x S_{\text{odd},x}}{2S_{\text{odd}}} - \frac{vS_{\text{odd},xx} + 2v_x S_{\text{odd},x}}{2S_{\text{odd}}} \\ &\quad + \frac{v_x S_{\text{odd},x} + vS_{\text{odd},xx}}{2S_{\text{odd}}} = 0. \end{aligned} \tag{3.27}$$

Since

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial x}(AS) - \frac{1}{2}A_{xx} \tag{3.28}$$

is well-known, what remains to be proved is

$$\frac{\partial}{\partial a}(AS - \frac{1}{2}A_x) = \frac{\partial}{\partial t}(vS - \frac{1}{2}v_x). \tag{3.29}$$

To show this, let us note that (2.1) and (3.1) entail

$$\begin{aligned} \frac{\partial}{\partial t}(vS_{\text{odd}}) &= A\frac{\partial}{\partial x}(vS_{\text{odd}}) + A_a S_{\text{odd}} \\ &= AS_{\text{odd},a} + A_a S_{\text{odd}} = \frac{\partial}{\partial a}(AS_{\text{odd}}). \end{aligned} \tag{3.30}$$

Using the relaltion $S_{\text{even}} = -S_{\text{odd},x}/(2S_{\text{odd}})$, together with (1.3), (2.1) and (3.1), we then find the following:

$$\begin{aligned} &\frac{\partial}{\partial t}(vS) - \frac{1}{2}v_{xt} - \frac{\partial}{\partial a}(AS) + \frac{1}{2}A_{xa} \\ &= \frac{vS_{\text{odd},x}(AS_{\text{odd}})_x}{2{S_{\text{odd}}}^2} - \frac{(vS_{\text{odd},x})_t}{2S_{\text{odd}}} - \frac{1}{2}v_{xt} \\ &\quad - \frac{AS_{\text{odd},x}(vS_{\text{odd}})_x}{2{S_{\text{odd}}}^2} + \frac{(AS_{\text{odd},x})_a}{2S_{\text{odd}}} + \frac{1}{2}A_{xa} \\ &= \frac{A_x S_{\text{odd},x}v}{2S_{\text{odd}}} - \frac{S_{\text{odd},x}v_t + (AS_{\text{odd}})_{xx}v}{2S_{\text{odd}}} - \frac{1}{2}v_{xt} \\ &\quad - \frac{AS_{\text{odd},x}v_x}{2S_{\text{odd}}} + \frac{A_a S_{\text{odd},x} + A(vS_{\text{odd}})_{xx}}{2S_{\text{odd}}} + \frac{1}{2}A_{xa} \\ &= \frac{S_{\text{odd},x}(A_x v - v_t - 2A_x v - Av_x + A_a + 2Av_x)}{2S_{\text{odd}}} \\ &\quad - \frac{A_{xx}v}{2} - \frac{1}{2}v_{xt} + \frac{Av_{xx}}{2} + \frac{1}{2}A_{xa} \\ &= \frac{1}{2}(A_x v - Av_x)_x - \frac{1}{2}A_{xx}v + \frac{1}{2}Av_{xx} = 0. \end{aligned} \tag{3.31}$$

Relations (3.26), (3.28) and (3.29) mean that ω is a closed form.

Q.E.D.

These propositions show that the verification of Ansatz 3.1 gives us at least locally, i.e., near $x = \lambda_0$, a 2-dimensional moduli space for (SL_{II}). In particular, we have really found a new deformation equation of (SL_{II}) for 0-parameter solution and pure 1-parameter solutions of (P_{II}).

We end this report by presenting the following

Conjecture 3.1. *Let us consider a pure solution λ_J of (P_J). Then the following should hold.*
(i) *For $J = \text{II}$, Ansatz 3.1 should be validated with the function v constructed in Section 2.*
(ii) *For $J = \text{II}$, a suitably chosen v should be expressed in the form (3.11) even when $\alpha_0\beta_0 \neq 0$.*
(iii) *For any J, deformation of (SL_J) with respect to each parameter contained in (P_J) should be possible. In addition, all such deformations together with the ordinary deformation with respect to t are compatible.*
(iv) *$E_{J,j}$ vanishes for any $j \geq 1$.*

References

[AKT] Aoki, T., T. Kawai and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. II. — Multiple-scale analysis of Painlevé transcendents. Structure of Solutions of Differential Equations, World Scientific, 1996, pp.1–49.

[KT1] Kawai, T. and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. I., Adv. in Math., **118** (1996), 1–33.

[KT2] ———: On the structure of Painlevé transcendents with a large parameter. II., Proc. Japan Acad. Ser. A, **72** (1996), 144–147.

[KT3] ———: WKB analysis of Painlevé transcendents with a large parameter. III. — Local equivalence of 2-parameter Painlevé transcendents. In prep.

[T] Takei, Y.: Article in preparation.